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# Different Orthocomplementations on the Subspace Lattice of a Finite-Dimensional Complex Vector Space†

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In this paper we prove, by using real closed fields and model theory, the following result: for any integer $n \geq 3$, there exist, on the lattice of all subspaces on the vector space $\mathbb{C}^n$, $2^{(2^{\aleph_0})}$ orthocomplementations leading to nonisomorphic structures of orthomodular lattices.

## 1. INTRODUCTION

In a Boolean algebra $B$, any element $a$ has a unique complement $a^\perp$ which is determined by the lattice structure of $B$: $a^\perp = \bigvee\{x \in B | x \wedge a = 0\}$. In other words, there exists at the most one structure of Boolean algebra on a distributive lattice. As orthomodular lattices (OMLs) are generalization of Boolean algebras, a question is: Do there exist on a bounded lattice two nonisomorphic structures of OMLs [6, Problem 27]?

The problem is of interest in lattice theory as well as in the logicoalgebraic approach to quantum mechanics because "the mathematical representation of the negative of any experimental proposition is the *orthogonal complement* of the mathematical representative of the proposition itself" [3].

Birkhoff solved this problem by considering on the linear space $\mathbb{Q}^4$ two scalar products leading to nonisomorphic OMLs of subspaces [2]. A more technical solution, using infinite-dimensional quadratic form theory, was given by Gross [5]. His method allows one to obtain a denumerable family of nonisomorphic OML structures on the same bounded lattice.

† This paper is dedicated to the memory of Fred Rüttimann.

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In this paper, we consider a related problem: Find a bounded lattice of infinite cardinal $\lambda$ carrying $2^\lambda$ nonisomorphic structures of OMLs (there exist at the most $2^\lambda$ nonisomorphic OMLs of cardinal $\lambda$). By using the theory of real closed fields and a result of S. Shelah related to the number on nonisomorphic models of an unstable theory, we prove that there exist on the lattice of all subspaces on the vector space $\mathbb{C}^n$, $n \geq 3$, $2^{(2^{\aleph_0})}$ orthocomplementations leading to nonisomorphic structures of OMLs.

The paper is organized as follows. In the next section, we characterize Hermitian forms defining isomorphic structures of OMLs on a vector space. Section 3 is devoted to real closed fields and complex Hermitian spaces. In the final section we prove the main result and give some concluding remarks.

## 2. ISOMORPHISMS OF OMLs OF SUBSPACES

If $E$ is a vector space over a field $K$, then $\mathcal{S}(E, K)$ denotes the modular lattice of all subspaces of $E$. Let $\sigma$ be an isomorphism from a field $K$ onto a field $K'$. A map $L$: $E \to E'$, where $E$ is a $K$-vector space and $E'$ a $K'$-vector space, is said to be $\sigma$-linear if, for $x, y \in E$ and $\lambda \in K$,

$$L(x + y) = L(x) + L(y), \qquad L(\lambda x) = \sigma(\lambda)L(x)$$

Any $\sigma$-linear map induces a map $\Phi_L$: $\mathcal{S}(E, K) \to \mathcal{S}(E', K')$ defined by $\Phi_L(M) = L(M) = \{L(x) | x \in M\}$. If $L$ is a bijection from $E$ onto $E'$, then $\Phi_L$ is an isomorphism between the lattices $\mathcal{S}(E, K)$ and $\mathcal{S}(E', K')$. All possible isomorphisms are obtained in this way if dim $E \geq 3$ [1; 8, Chapter 2, §2].

If $E$ is an infinite-dimensional vector space, then there exists no orthocomplementation on $\mathcal{S}(E, K)$ [1, Theorem 2, p. 111] and if if the dimension of $E$ is finite and not less than 3, then the orthocomplementations on $\mathcal{S}(E, K)$ are induced by some special semibilinear forms. We expand this point.

Let $\gamma$ be an involutary antiautomorphism of the field $K$. A $\gamma$-Hermitian form is a map $\langle \cdot, \cdot \rangle$: $E^2 \to K$ with the following properties:

- $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$ and $\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$, $(x, y, x_1, x_2, y_1, y_2 \in E)$.
- $\langle \lambda x, \mu y \rangle = \lambda \langle x, y \rangle \gamma(\mu)$ $(x, y \in E$ and $\lambda, \mu \in K)$.
- $\langle x, y \rangle = \gamma(\langle y, x \rangle)$ $(x, y \in E)$.

The form is said to be defined if $\langle x, x \rangle = 0$ implies $x = 0$.

*Example.* Let $\gamma$ be an involutary antiautomorphism of a field $K$. For any integer $n > 0$, the map

$$((x_i)_{1 \leq i \leq n}, (y_i)_{1 \leq i \leq n}) \to \sum_{i=1}^{n} x_i \gamma(y_i)$$

is a $\gamma$-Hermitian form on the vector space $K^n$. Remark that this form is defined if and only if $\Sigma_{i=1}^n x_i \gamma(x_i) = 0$ implies $x_i = 0$, $i \in [1, n]$.

*Proposition 1 [3].* Let $E$ be a left vector space on a field $K$ with $3 \leq \dim E < \infty$. The orthocomplementations of the lattice $\mathcal{S}(E, K)$ are all the maps

$$F \in \mathcal{S}(E, K) \to F^{\perp} = \{x \in E | \langle x, F \rangle = 0\} \in \mathcal{S}(E, K)$$

where $\langle \cdot, \cdot \rangle$ is a defined $\gamma$-Hermitian form on $E^2$. A $\gamma$-Hermitian form $\langle \cdot, \cdot \rangle$ and a $\gamma'$-Hermitian form $\langle \cdot, \cdot \rangle'$ induce the same orthocomplementation if and only if there exists $k \in K$, $k \neq 0$, such that for all $x, y \in E$, $\lambda \in K$,

$$\langle x, y \rangle' = \langle x, y \rangle \cdot k, \qquad \gamma'(\lambda) = k^{-1} \cdot \gamma(\lambda) \cdot k$$

This proposition allows us to characterize Hermitian forms leading to isomorphic OMLs of subspaces.

*Proposition 2.* Let $\langle \cdot, \cdot \rangle$ be a $\gamma$-Hermitian form, $\langle \cdot, \cdot \rangle'$ be a $\gamma'$-Hermitian form on the $K$-vector space $E$, with $3 \leq \dim E < \infty$, and let $\perp$, $\perp'$ be the corresponding orthocomplementations on $\mathcal{S}(E, K)$. The orthocomplemented lattices $(\mathcal{S}(E, K), \perp)$ and $(\mathcal{S}(E, K), \perp')$ are isomorphic if and only if there exists a $\sigma$-isomorphism $L: E \to E$ and $k \in K$, $k \neq 0$, such that

$$\langle L(x), L(y) \rangle' = \sigma(\langle x, y \rangle) \cdot k, \quad \sigma^{-1}\gamma'\sigma(\lambda) = k^{-1} \cdot \gamma(\lambda) \cdot k$$
$$(x, y \in E, \lambda \in K) \tag{1}$$

Remark that relations (1) become

$$\langle L(x), L(y) \rangle' = k \cdot \sigma(\langle x, y \rangle), \qquad \sigma^{-1} \circ \gamma' \circ \sigma = \gamma$$

if the field $K$ is commutative.

*Definition 1.* Two defined Hermitian forms on a $K$-vector space $E$ are said to be *orthoequivalent* if they yield to isomorphic structures of OML on $\mathcal{S}(E, K)$.

Now, if we want to find a vector space $K^n$ with a maximal number of nonisomorphic structures of OMLs on its lattice of subspaces, we are confronted with the following:

1. Find a field $K$ with an abundance of involutary automorphisms.
2. Characterize in different ways Hermitian forms which are orthoequivalent.

The field of complex numbers has $2^{(2^{\aleph_0})}$ involutary automorphisms and so it seems appropriate to use complex vector spaces. The next proposition will provide a solution to the second question.

If $f$ is an automorphism of a field $K$, then we denote by $F_f$ the fixed field of $F$: $F_f = \{x \in K | f(x) = x\}$.

*Proposition 3.* If a $\gamma$-Hermitian form and a $\gamma'$-Hermitian form on the same vector space $E$ over a commutative field $K$, with $3 \leq \dim E \leq \infty$, are orthoequivalent, then the fixed fields $F_\gamma$ and $F_{\gamma'}$ are isomorphic.

*Proof.* If the two Hermitian forms are orthoequivalent, then, by Proposition 2, there exists an automorphism $\sigma$ of $K$ such that $\sigma^{-1} \circ \gamma' \circ \sigma = \gamma$. If $x \in F_\gamma$, then $(\gamma' \circ \sigma)(x) = (\sigma \circ \gamma)(x) = \sigma(x)$ and so $\sigma(x) \in F_{\gamma'}$. Conversely, if $\sigma(x) \in F_{\gamma'}$, then $(\gamma' \circ \sigma)(x) = \sigma(x)$ and $(\sigma^{-1} \circ \gamma' \circ \sigma)(x) = x$. As $\sigma^{-1} \circ \gamma' \circ \sigma = \gamma$, $\gamma(x) = x$, which proves that $x \in F_\gamma$. So $\sigma(F_\gamma) = F_{\gamma'}$ and the fixed fields of $\gamma$ and $\gamma'$ are isomorphic.

In the next section we will characterize fixed fields of involutary automorphisms of $\mathbb{C}$. For this aim, it is useful to introduce, more generally, real closed fields.

## 3. REAL CLOSED FIELDS

A commutative field $K$ is said to be real closed if it fulfills one of the following equivalent statements:

1. The characteristic of $K$ is 0, $-1$ has no square root in $K$, and $K[i]$ is algebraically closed (i.e., $K[X]/X^2 + 1$ is algebraically closed).
2. $K$ is an ordered field and no algebraic extension of $K$ can be ordered.
3. $K$ is an ordered field in which polynomials satisfy the condition that, if $P(X) \in K[X]$, $a, b \in K$, and $P(a)P(b) < 0$, then there exists $c \in K$ between $a$ and $b$ such that $P(c) = 0$.

By using the third statement, it is easy to check that real closed fields constitute a first-order theory in a language obtained by adding to the language of fields a binary relation symbol corresponding to the order relation.

Real closed fields are also closely related to fixed fields of involutary automorphisms of algebraically closed fields, and in the case of complex numbers we have the following result.

*Proposition 4.* The correspondence

$$\gamma \to F_\gamma = \{x \in \mathbb{C} | \gamma(x) = x\}$$

maps the set of all involutary automorphisms of $\mathbb{C}$, different from the identity, bijectively onto the set of all real closed subfields $F$ of $\mathbb{C}$ such that $[\mathbb{C} : F] = 2$.

*Abridged Proof.* If $\gamma$ is an involutary automorphism of $\mathbb{C}$, different from the identity, then the automorphism group generated by $\gamma$ is of order two and by the theorem of Artin, $[\mathbb{C} : F_\gamma] = 2$. As $i \notin F_\gamma$, $\mathbb{C} = F_\gamma[i]$ and $F_\gamma$ is a real closed field. Conversely, if $[\mathbb{C} : F] = 2$, then $\mathbb{C} = F + iF$ and $\gamma : \mathbb{C} \to \mathbb{C}$ defined by $\gamma(x + iy) = x - iy$ is an involutary automorphism of $\mathbb{C}$, different from the identity, and such that $F = F_\gamma$.

*Proposition 5.* Let $\gamma$ be an involutary automorphism of $\mathbb{C}$, different from the identity. The $\gamma$-Hermitian form on $\mathbb{C}^n$

$$((x_i)_{1\le i\le n}, (y_i)_{1\le i\le n}) \to \sum_{i=1}^{n} x_i\gamma(\gamma_i)$$

is defined.

*Proof.* Let $y$ be a square root of the complex number $x$. We have

$$x\gamma(x) = y^2\gamma(y^2) = [y\gamma(y)]^2$$

Moreover, $\gamma[y\gamma(y)] = y\gamma(y)$ and so $y\gamma(y) \in F_\gamma$. Therefore, for any $x \in \mathbb{C}$, $x\gamma(x)$ is a square of an element of the real closed field $F_\gamma$ which is an ordered field. Thus $\Sigma_{i=1}^n x_i\gamma(x_i)$ is a sum of squares of elements of $F_\gamma$ and $\Sigma_{i=1}^n x_i\gamma(x_i) = 0$ implies $x_i = 0$, $i \in [1, n]$. The form is defined.

Now the problem is to determine the number of nonisomorphic real closed subfields $F$ of $\mathbb{C}$ such that $[\mathbb{C} : F] = 2$. A result from model theory will be useful.

## 4. THE MAIN RESULT

In ref. 7 the following definition is a characterization of an unstable theory.

*Definition 2 [7].* A first-order theory $T$ in a language $L$ is unstable if $T$ has a model $M$ and if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of $M$ and a formula $\phi(x, y)$ of $L$ such that, for every $m$, $l$,

$$M \models \phi(a_m, a_l) \Leftrightarrow m \le l$$

It is obvious that the theory of real closed fields is unstable: $\mathbb{R}$ is a model of this theory and if $\phi(x, y)$ is $x \le y$ and $a_n = n$, then $\mathbb{R} \models \phi(a_m, a_l) \Leftrightarrow m \le l$.

The main result of ref. 7 states that an unstable theory possesses in any uncountable cardinal a maximal number of models; precisely:

*Proposition 6 [7].* If $T$ is an unstable first-order theory in a denumerable language, then $T$ has $2^\lambda$ nonisomorphic models in every uncountable cardinal $\lambda$.

*Theorem 1.* For any integer $n \ge 3$, there exist $2^{(2^{\aleph_0})}$ nonisomorphic structures of OMLs on the lattice of all subspaces of the vector space $\mathbb{C}^n$.

By Proposition 6, there exists a family $(F_k)_{k<2^{(2^{\aleph_0})}}$ of pairwise nonisomorphic real closed fields of cardinal $2^{\aleph_0}$. As all the algebraically closed fields of cardinal $2^{\aleph_0}$ are isomorphic, let $f_k$ be an isomorphism from the algebraically closed field $F_k[i]$ onto $\mathbb{C}$. Since field isomorphisms between real closed fields are also order isomorphisms, the real closed fields $f_k(F_k)$ constitute a set of $2^{(2^{\aleph_0})}$ pairwise nonisomorphic real closed subfields of $\mathbb{C}$ satisfying $\mathbb{C} = f_k(F_k)[i]$. Propositions 3 and 5 complete the proof.

## 5. CONCLUDING REMARKS

1. Theorem 1 can be generalized to any infinite cardinal, but the proof is different in the countable case. It can be also generalized to structures related to OMLs: Baer; ∗-rings, orthosymmetric ortholattices, . . . [4].

2. The lattices of Theorem 1 are irreducible, modular, and of finite height. By using direct products or pastings, one can obtain other lattices carrying a maximal number of nonisomorphic structures of OMLs.

3. All the lattices with different orthomodular orthocomplementations are infinite. Does there exist a finite lattice with different orthomodular orthocomplementations?

4. Do, for $n$ fixed, all the nonisomorphic OMLs of Theorem 1 satisfy the same equations or the same first-order sentences?

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